USING THE SPLINE FUNCTIONS FOR THE CALCULATION OF LIMNIMETRICS KEY

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Abstract

The spline interpolation presume to find a set of polynomials of first degree (linear), second degree (parabolic), third degree (cubic), which are defined on an interval, and their connection on the ends of intervals makes, by imposing that these functions has to be continuous on point, to be smooth and without the inflections points. Mathematically presume as their derivatives of first and second order to the left and to the right of point to be in their turn and then continuous. Considering as the limnimetric key (dependence on the level of a river section in relation to the discharge which flow through section) is a smooth curve end with continuous growth (all ensemble that define the general curve must be monotonous), impose the additional conditions on monotony and flexion polynomials and in their interval by operating, not only in connection node of interpolation.

If the first derivative of a polynomial is positive on an interval and if the interval which operate the spline polynomial with positive variation of the first derivative, then the spline function in that interval is monotone increasing.

If the second derivative of a polynomial spline in the interval of definition a function does not change of sign, and the spline polynomial of interval not present the inflexions, then, the second derivative of a polynomial spline has not the roots in within the interval in which it operates.

Key words: Spline cubic, limnimetric key.

INTRODUCTION

The paper propose to realize an algorithm for calculation using the polynomials of third degree for simulation by numerical calculation of a continuous curve, monotone, increasing end smooth by connection of polynomials of third degree equivalency with the limnimetric key.

The method can be applied as input data for different engineering application e.g. hydrological events, floods, sediment transport, floodplain reinforcement (Sandu, 2015; Vîrsta 2007).

MATERIALS AND METHODS

The method for calculation of limnimetric key based on the interpolation, and the date used in calculation are the measurement of flow and levels by the hydraulic method Chezy for the studied section.

- Deduction the polynomials by spline cubic interpolation

The polynomials by spline cubic interpolation by order III or the cubic interpolation have the next expression:

\[ S_{III}^j(x) = y_i + m_i (x - x_i) + q_i (x - x_i)^2 + h_i (x - x_i)^3 \]  

(1)

Where \( x_i \) is the end from left of the polynomial spline cubic which operate in the interval \( (x_{i-1} \, , \, x_i) \).

Impose polynomial by interpolation (1) the condition by continuity on a current node of interpolation (Gogonea, 1998), thus:

\[
S_{III}^j(x_{i+1}) = y_i + m_i (x_{i+1} - x_i) + q_i (x_{i+1} - x_i)^2 + h_i (x_{i+1} - x_i)^3 \\
S_{III}^j(x_{i+1}) = y_{i+1} + m_{i+1} (x_{i+1} - x_{i}) + q_{i+1} (x_{i+1} - x_{i})^2 + h_{i+1} (x_{i+1} - x_{i})^3 \\
S_{III}^j(x_{i+1}) = S_{III}^j(x_{i+1}) = S_{III}^j_{x_{i+1}}(x_{i+1})
\]
Combining the relations above and performing the elementary calculation it results:

\[ y_{i+1} = y_i + m_i \left( x_{i+1} - x_i \right) + b_i \left( x_{i+1} - x_i \right)^2 + b_i \left( x_{i+1} - x_i \right)^3 \]  

(2)

Impose as the polynomials by interpolation to be smoothed which means as the first derivate of cubic polynomials by interpolation on the current node at left i, respectively to the right i+1, must be continuous, so equal on the node of interpolation (Pavaloiu, 1981).

The derivative of the first order, general on intervals of interpolation it obtain by derivation in relation with the variable x of general function of interpolation given of formula (1)

\[ S_{III}^i (x) = m_i + 2a_i \left( x - x_i \right) + 3b_i \left( x - x_i \right)^2 \]  

(3)

It impose the condition of continuity these derivatives on node of interpolation what mathematical it write thus:

\[ \left[ S_{III}^i \right]' (x_{i+1}) = \left[ S_{III}^i \right]' (x_i) \]

and by replacement in formula (3) get:

\[ S_{III}^i (x_{i+1}) = m_i + 2a_i \left( x_{i+1} - x_i \right) + 3b_i \left( x_{i+1} - x_i \right)^2 \]

\[ S_{III}^i (x_{i+1}) + 1 = m_i + 1 + 2a_i \left( x_{i+1} - x_i \right) + 3b_i \left( x_{i+1} - x_i \right)^2 \]

by simple algebraic processing and imposing the condition of continuity to derivative on node of interpolation get:

\[ m_{i+1} = m_i + 2a_i \left( x_{i+1} - x_i \right) + 3b_i \left( x_{i+1} - x_i \right)^2 \]  

(4)

As shown the previous result, the algebraic system of calculation for the parameters spline cubic polynomials will have form (7) and spline functions will be given by formula (8):

\[ y_{i+1} = y_i + m_i \left( x_{i+1} - x_i \right) + b_i \left( x_{i+1} - x_i \right)^2 + b_i \left( x_{i+1} - x_i \right)^3 \]

(7)

\[ y = y_i + m_i \left( x - x_i \right) + a_i \left( x - x_i \right)^2 + b_i \left( x - x_i \right)^3 \]  

(8)

- **Determination of the additional conditions for the monotony of spline cubic**

Additionally, imposing the conditions as the spline cubic polynomials to be monotonous increasing, do not present the point of inflexion or return inside of operating, what we lead with the thinking to study the sign first and the second derivative inside the intervals where operating the spline cubic polynomials (Agopian et al., 1963).

The analysis of the sign first derivative it makes processing the relation (8) by dissolution the parenthesis and regrouping the terms, it makes next function of second degree.

\[ \frac{\partial y}{\partial x} = \left[ S_{III}^i \right]' (x) = 3b_i x^2 + 2 \left( a_i - 3b_i x_i \right) x + m_i - 2a_i x_i + 3b_i x_i^2 \]  

(9)
To study the sign of this derivative, we need the discriminant of the equation given by relation (9) and the roots of this equation:

\[
\Delta = 4 \left( a_i - 3 b_i x_i \right)^2 - 4 \times 3 b_i \left( m_i - 2 a_i x_i + 3 b_i x_i^2 \right)
\]

Processing the relation expression discriminant get the expression:

\[
\Delta = 4 \left( c_i^2 - 3 b_i m_i \right)
\]  

(10)

Discussion:
If the discriminant is negative, than the sign of relation (10) is the sign his b_i on the entire real axis (Table 1).
If the discriminant is positive, than the sign relation (10) is same with b_i outside the roots and contrary his b_i between roots (Table 2).

Forwards we study the sign of the second derivative of the function of interpolation; this can get by derivation in report with the variable x of relation (9):

\[
\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial y}{\partial x} \right) = 6 b_i x + 2 a_i - 6 b_i x_i
\]

(11)

The equation attached the relation (11) is:

\[
3 b_i x + a_i - 3 b_i x_i = 0
\]

(12)

At left the root have the contrary sign of b_i, and at right the same right with b_i (Table 1 and Table 2).

Applying Rolle's Theorem it has the next context of discussion the variation and monotony the spline cubic monotony inside the interval of operating.

Table 1. The table of variation and monotony the spline cubic polynomials when the discriminant is smaller than zero.

| \( \Delta = 0 \) | \( x \) | \( -\infty \) | \( x_i - \alpha_i \) | \( \infty \) |
|-----------------|--------|----------------|------------------|
| \( \frac{\partial y}{\partial x} \) | the same sign b | \( \alpha_i \) | \( x_i - \alpha_i \) |
| \( \frac{\partial^2 y}{\partial x^2} \) | the contrary sign b | 0 | the same sign b |

If the discriminant is smaller than zero, the function of interpolation is strict increasing and without the point of inflexion if b_i is positive on entire domain of definition of the spline cubic polynomials, interpretation Table 1.

Table 2. The table of variation and monotony the spline cubic polynomials when the discriminant is bigger than zero.

<table>
<thead>
<tr>
<th>( \Delta &gt; 0 )</th>
<th>( x )</th>
<th>( x_i - \alpha_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\partial y}{\partial x} )</td>
<td>( x_i - \alpha_i )</td>
<td>( x_i - \alpha_i )</td>
</tr>
<tr>
<td>( \frac{\partial^2 y}{\partial x^2} )</td>
<td>0</td>
<td>the same sign b</td>
</tr>
</tbody>
</table>

If the root of the second derivative is between the roots of first derivative, then \( \alpha_i > 0 \) and, in addition, \( x_i \leq \left[ x_i' \right]_0 + \alpha_i \) (Table 2).
If the root of the second derivative is smaller than those of the first derivative, the sign will be dictated by the entire root, thus being available the condition above (Table 2).

Observation, the roots equation of the first derivative is given by the formulas:

\[
\left[ x_1' \right]_0 = \frac{-2 \left( a_i - 3 x_i b_i \right) \pm \sqrt{\Delta}}{6 b_i}
\]

(13)

RESULTS AND DISCUSSIONS

The hydrological practice has shown that the application of numerical calculation for elaboration the limnmetric keys leaving from the direct interpolation is very laborious, and the volume of calculation presumes the computers with the memory enough of big, especially the memory of ram and speed of processing.
It can approach the problem using direct for the node of interpolation the fields of points result of direct product of value sectional area section with speed, respectively, the correspondents for different values of level. This way we are using the spline cubic polynomials, imposing the conditions of slope for the curves of interpolation at beginning and end of interval, and also the conditions of monotony, smoothness and inflection inside the intervals of operating these functions spline cubic.
Forwards we present the Table 3 with the parameters the functions by spline interpolation and the criteria what must fulfilled for monotony, smoothness and inflection of the spline cubic function on the interval of operating.
Table 3. The parameters of the spline cubic interpolation functions $H_i(Q)$

<table>
<thead>
<tr>
<th>node</th>
<th>level</th>
<th>m</th>
<th>a</th>
<th>h</th>
<th>A</th>
<th>x</th>
<th>y</th>
<th>x'</th>
<th>y'</th>
<th>x''</th>
<th>y''</th>
</tr>
</thead>
<tbody>
<tr>
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<td>2</td>
<td>0.5</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>-1</td>
<td>5</td>
<td>0</td>
<td>-3</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>-1</td>
<td>1</td>
<td>2</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
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<td>4</td>
<td>2</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>5</td>
<td>-5</td>
<td>0</td>
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<td>0</td>
<td>0</td>
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<td>0</td>
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<td>1</td>
<td>2</td>
<td>-1</td>
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<td>-5</td>
<td>0</td>
<td>0</td>
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<td>0</td>
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<td>-1</td>
<td>2</td>
<td>-1</td>
<td>0</td>
<td>-3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Figure 1. The limnimetric key using the spline parabolic interpolation

Figure 2. The percentage error between the real flows results of measurements and the flows result by calculation-the spline cubic method

CONCLUSIONS

The limnimetric key is a curve formed by several polynomials of interpolation of the same type which are defined on an interval or the set of points (the nodes of interpolation) what must to respect the conditions:

- The polynomials of interpolation and their derivatives of order one and two to the left and right on point of interpolation must be continuous.
- The polynomials of interpolation must be monotonous increasing, smooth, do not present the inflections and return in inside the interval of operating of function of interpolation.

REFERENCES